Chapter 6
State Estimation
Part 3
6.3 Inertial Navigation Systems
Outline

• 6.3 Sensors for State Estimation
  – 6.3.1 Introduction
  – 6.3.2 Mathematics of Inertial Navigation
  – 6.3.3 Errors and Aiding in Inertial Navigation
  – 6.3.4 Example: Simple Odometry Aided AHRS
  – Summary
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History

• Historical roots in German Peenemunde Group.
• Modern form credited to Charles Draper et al. @MIT.
• 1940s Germany:
  – V2 program, gyroscopic guidance
• 1950s Draper Labs, MIT:
  – Shuler tuned INS
  – Floated rate integrating gyros (0.01 deg/hr)
• 1960s DTGs
  – not floated or temp compensated
• 1970s RLGs, USA
• 1980s Strapdown INS
• 1990s GPS
Introduction

• Advantages
  – Most accurate dead reckoning available.
  – Useful in wide excursion (outdoor) missions.
  – Work anywhere where gravity is known.
  – Are jamproof - require no external information.
  – Radiate nothing - exhibit perfect stealth.

• Disadvantages
  – Cannot sense accelerations of unpowered space flight.
  – Most errors exhibit Schuler oscillation (advantage?).
  – Most errors are time dependent.
  – Requires input of initial conditions.
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6.3.2 Mathematics of Inertial Navigation
(Concept)

- Use Inertial Properties of Matter
  - Accelerometers
  - Gyros

- Do “Dead Reckoning”
  - Integrate acceleration twice

\[
\begin{align*}
\dot{\mathbf{v}}(t) &= \mathbf{v}(0) + \int_{0}^{t} \mathbf{a} \, dt \\
\dot{\mathbf{r}}(t) &= \mathbf{r}(0) + \int_{0}^{t} \mathbf{v} \, dt
\end{align*}
\]
6.3.2 Mathematics of Inertial Navigation

(Naïve Concept)

• Just integrating 3 accels will not work for a lot of reasons:
  – Accelerometers measure wrong quantity.
  – They measure it in wrong reference frame.
  – They represent it in wrong coordinate system.

• The quest for ever better engineering solutions to these problems is the primary reason for the complexity of the modern INS.
6.3.2 Problem 1: Equivalence

- Accelerometers don’t measure acceleration.
- Specific force is: \( \vec{f} = \hat{a} - \hat{g} \)
- Fix: must know gravity, then: \( \hat{a} = \vec{f} + \hat{g} \)

![Diagram showing different states of acceleration: Freefall (Space), At Rest (Earth), Accelerating, Both](image)
Problem 2: Inertial Frame of Reference

• Now have inertial acceleration.
  – Want earth-referenced acceleration.

• Fix: account for earth angular velocity:
  – “Apparent forces”.
  – “gravity”, not gravitation
Problem 3: Body Coordinates

• Accelerometers are fixed to vehicle.
  – Want to integrate in the world frame.
• Need to know instantaneous heading.
• So..., track orientation
  – Use gyros.

\[ \theta(t) = \int_{0}^{t} \omega(t) \, dt \]
6.3.2.1 First Fix: Specific Force to Acceleration

• We know specific force is not acceleration.

• The fundamental equation of inertial navigation is Newton’s 2nd law applied to the accelerometers:

\[ \sum F = \vec{T} + \vec{W} = m\vec{a}^i \]

• Need to solve for acceleration....
6.3.2.1 First Fix: Specific Force to Acceleration

• Solving for acceleration:

\[ \vec{a}^i = \frac{\vec{T}}{m} + \frac{\vec{W}}{m} = \vec{u} + \vec{w} \]

Gravitational field

• Note: you need to know the gravitational field anywhere you want to do inertial navigation.
6.3.2.2 Second Fix: Remove Apparent Forces

- Moving vehicle is a moving reference frame.
  - Hence, sensors on-board will sense apparent forces.
  - Remove them with Coriolis law.
6.3.2.2 Second Fix: Remove Apparent Forces

- Define Frames:
  - i: “inertial”, geocentric, nonrotating.
  - e: “earth”, geocentric, rotating.
  - v: “vehicle”, fixed to accels. Also known as body frame.
6.3.2.2 Second Fix: Remove Apparent Forces

• Define:

\[
\begin{align*}
\mathbf{r}^x_v & \quad \text{Position of vehicle measured in frame } x \\
\mathbf{v}^x_v & \quad \text{Velocity of vehicle measured in frame } x \\
\mathbf{a}^x_v & \quad \text{Acceleration of vehicle measured in frame } x
\end{align*}
\]
6.3.2.2 Second Fix: Remove Apparent Forces

- Basic acceleration transformation under negligible angular acceleration:

\[
\ddot{a}_o = \ddot{a}_m + \ddot{f} + 2 \omega_m \times v_o + \omega_m \times (\omega_m \times r_o)
\]

- Let “o” = v, “m” = e, and “f” = i:

\[
\ddot{a}_v = \ddot{a}_e + \ddot{a}_i + 2 \omega_e \times v_v + \omega_e \times (\omega_e \times r_v)
\]

- The i and e origins are coincident. Hence:

\[
\ddot{a}_e = 0
\]
6.3.2.2 Second Fix: Remove Apparent Forces

- Also, let the earth sidereal rate be given by:

  \[ \omega_e = \Omega \]

- Now, moving the earth acceleration to the left hand side, we have:

  \[ \dot{a}_v = a_v - 2\Omega \times v_v - \dot{\Omega} \times (\Omega \times r_v^e) \]

- Substituting for specific force:

  \[ \dot{a}_v = \dot{t} - 2\Omega \times v_v + \dot{\omega} - \dot{\Omega} \times (\Omega \times r_v^e) \]
6.3.2.2 Second Fix: Remove Apparent Forces

- The quantity: \[ \ddot{\mathbf{w}} - \dot{\mathbf{\Omega}} \times (\mathbf{\Omega} \times \mathbf{r}_v^e) \]

- Is known as “gravity” and denoted \( \ddot{\mathbf{g}} \)

- Finally, we have “the” equation of inertial navigation.

\[
\dot{a}_v = \frac{d}{dt} (v_v^e) = \ddot{t} - 2\dot{\mathbf{\Omega}} \times \mathbf{v}_v^e + \ddot{\mathbf{g}}
\]

This is the derivative of the velocity relative to e as computed by an earth-fixed observer.

Specific Force

Coriolis

Gravity
6.3.2.2 Second Fix: Remove Apparent Forces

- The computed solution in coordinate system independent form is:

\[
\vec{v}_v^e = \int_{t_0}^{t} [\dot{t} - 2\Omega \times \vec{v}_v^e + \vec{g}] dt|_e + \vec{v}_v^e(t_0)
\]

\[
\vec{r}_v^e = \int_{0}^{t} \vec{v}_v^e dt|_e + \vec{r}_v^e(t_0)
\]

- These are only valid if you integrate in the earth frame (i.e. in earth-fixed coordinates).

You need to know:
- a model of gravity
- earth sidereal rate
- specific forces
- initial position
- initial velocity
- (gyros don’t appear in vector form)
6.3.2.2.1 Vector Formulation

\[ \dot{t} \quad \text{from} \quad \text{accels} \]

\[ \frac{-GM_e(t)}{(\cdot)^3} \quad \text{gravitation} \]

\[ \tilde{\Omega} \times \tilde{\Omega} \times \] (\text{centrifugal})

\[ 2\tilde{\Omega} \times \] (Coriolis)

\[ \int_0^t dt \]

\[ \vec{v}_e(t_0) \quad \vec{v}_e \]

\[ \int_0^t dt \]

\[ \vec{r}_v(t_0) \quad \vec{r}_v \]
6.3.2.2 Gravity and Gravitation

- Gravity is the force per unit mass required to fix an object wrt the Earth. It **includes centrifugal force**.
- Gravitation is the force described in Newton’s law of gravitation.

\[
\vec{W} = \vec{w} = \frac{-GM_e e^e}{|e^e| 3 \hat{v}_v} \nabla
\]

- Only at the equator and at the poles does gravity point toward the center of the earth.
6.3.2.3 Third Fix: Adopt a Coordinate System

• The heart of the INS is the inertial measurement unit (IMU) containing 3 accelerometers and 3 gyros.

• The gyros are used to track the orientation of the vehicle wrt the earth.

• You need orientation because:
  – \( \hat{g} \) and \( \hat{\Omega} \) are known in earth coordinates, whereas….  
  – \( \hat{t} \) and \( \hat{\omega} \) are measured in body coordinates in a modern strapdown system.

• Can’t add em up unless they are in the same coordinate system.
6.3.2.3.1 Third Fix: Euler Angles

• Step 1: Integrate the gyros:

\[
\begin{bmatrix}
\theta \\
\phi \\
\psi
\end{bmatrix}
= \int \begin{bmatrix}
\dot{\theta} \\
\dot{\phi} \\
\dot{\psi}
\end{bmatrix} dt +
\begin{bmatrix}
\theta(0) \\
\phi(0) \\
\psi(0)
\end{bmatrix}
= \int \begin{bmatrix}
c\phi & 0 & s\phi \\
t\theta s\phi & 1 & -t\theta c\phi \\
-s\phi & 0 & c\phi \\
c\theta & 0 & c\theta
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{bmatrix} dt +
\begin{bmatrix}
\theta \\
\phi \\
\psi
\end{bmatrix}(0)
\]
6.3.2.3.2 Third Fix: Direction Cosines

- Step 1: Or, use direction cosine form (better):

\[
\delta_\Theta = \omega \delta t \quad \delta \Theta = \left| \delta \Theta \right|
\]

\[
f_1(\delta \Theta) = \frac{\sin \delta \Theta}{\delta \Theta} \quad f_2(\delta \Theta) = \frac{(1 - \cos \delta \Theta)}{\delta \Theta^2}
\]

\[
R_{k+1}^k = I + f_1(\delta \Theta)[\delta_\Theta]^X + f_1(\delta \Theta)([\delta_\Theta]^X)^2
\]

\[
R_{n+1}^n = R_k^n R_{k+1}^k
\]
6.3.2.3.3 Third Fix: Quaternions

- Step 1: Or, use the quaternion form (best):

\[
\delta \Theta = \omega \delta t \\
\tilde{q}_{k+1}^k = \cos \delta \Theta [I] + \sin \delta \Theta \left[ \left( \times [\tilde{\omega}_b] \right) / |\tilde{\omega}_b| \right] \\
\tilde{q}_{k+1}^n = \tilde{q}_{k+1}^k \tilde{q}_k^n
\]
When orientation aiding is rare (yaw aiding is typically rare), it may be useful to remove earth rate from the gyros:

\[ v_\omega^e = v_\omega^i - R^v_e R^e_i \Omega^i \]

Or its projection onto the yaw axis will be integrated.

Where is this projection greatest?

Note: \( n = e \) here
6.3.2.3 Third Fix: Adopt a Coordinate System

- Step 2: Integrate the accels:

\[
\dot{v}^e_v = \int_0^t [R_v^e t + g - 2\Omega \times v^e_v] dt + v^e_v(t_0)
\]

- Step 3: Integrate the velocity:

\[
r^e_v = \int_0^t v^e_v dt + r^e_v(t_0)
\]
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6.3.3.1 Sensitivity

**TABLE 3. Term Magnitudes**

<table>
<thead>
<tr>
<th>Term Name</th>
<th>Expression</th>
<th>Nominal Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Force</td>
<td>$\ddot{r}$</td>
<td>0.1 g</td>
</tr>
<tr>
<td>Gravitational</td>
<td>$\ddot{g}$</td>
<td>1.0 g</td>
</tr>
<tr>
<td>Centrifugal</td>
<td>$\Omega \times \Omega \times \dot{r}_v^e$</td>
<td>3.5x10^{-3} g</td>
</tr>
<tr>
<td>Coriolis</td>
<td>$2\Omega \times \dot{v}_v^e$</td>
<td>1.5x10^{-4} g</td>
</tr>
</tbody>
</table>

- Acceleration is multiplied by the square of time.
  - $1 \text{ hour}^2 = 13 \text{ million} \text{ secs}^2$.
- After 1 hour, the Coriolis (smallest) term accounts for over 9.5 Km of error.

For a vehicle at the equator, moving eastward at a velocity of 10 meters per second, and accelerating at 0.1 g
Error Explosion

• For a 10 m/s vehicle at the equator, the Coriolis term is tiny:
  – $1.5 \times 10^{-4}$ g

• Consider an error of this magnitude...

• In one hour:
  – $t^2 = (3600)^2 = 13$ million !!

• Position Error:
  – 9.5 Kilometers!!!
Error Dynamics: Gravity Feedback

- Consider predictions of gravity direction based on position.
- This is called a Shuler loop.

Step 1: Orientation error, (system thinks it is level).

Step 2: Spring is Deflected this way.

Step 3: Interpret as Motion this way.

Step 4: Which rotates gravity prediction until more motion is unnecessary.
Here
Perturbative Analysis

• If the accelerometer biases are constant, the solutions are:

\[
\delta x = \frac{\delta t_x}{g_0/R_0} \left[ 1 - \cos \left( \sqrt{\frac{g_0}{R_0}} t \right) \right]
\]

\[
\delta y = \frac{\delta t_y}{g_0/R_0} \left[ 1 - \sin \left( \sqrt{\frac{g_0}{R_0}} t \right) \right]
\]

\[
\delta z = \frac{\delta t_z}{2g_0/R_0} \left[ \cosh \left( \sqrt{\frac{2g_0}{R_0}} t \right) \right]
\]

• Gravity field is a mixed blessing!!
6.3.3.2 Aided Inertial Mode

- Used on mobile robots:
  - Zero velocity update
  - Odometry
  - GPS
  - Landmarks / Map matching
  - Magnetic heading

- Used more generally:
  - Barometric altitude
  - Radar altimeters
  - Doppler radar velocity

Note: Net effect of velocity aiding is to convert error dynamics from that of free Inertial to that of odometry.
6.3.3.3 Initialization

- In **self alignment**, the INS is left stationary and:
  - Accels determine direction of gravity in process called **levelling**.
  - Gyros determine direction of earth’s spin vector in a process called **gyrocompassing**.
  - Latitude can also be estimated in this way but not longitude.

- Modern GPS aided systems do “moving base alignment” where the difference in GPS readings over time can be used to determine vehicle heading.
Initialization

• Need to measure two non-collinear vectors.
• Earth conveniently has two:
  – Gravity - easy
  – Earth spin – takes time, several minutes
  – Angle between them gives latitude.
• Gives orientation wrt earth and latitude.
Smiths Industries INS

• Without GPS
  – Static Heading: <0.1 deg. rms
  – Position: <0.35% DT Horizontal
  – Altitude: <0.25% DT Vertical
• With GPS
  – Dynamic Heading: <0.1 deg. rms
  – Position: <10 meters CEP
  – Altitude Accuracy: <10 meters VEP
• Pitch and Roll Outputs: <0.05 deg. rms

• Initialization Time – Static: 3-5 minutes (gyrocompassing)
• Initialization Time – On-the-Move: 1-3 minutes
Watson Industries AHRS E304

- **Attitude:**
  - 0.25% static, 2% dynamic

- **Heading:**
  - 1% static, 2% dynamic

- **Angular Rate:**
  - Scale factor 1%
  - Bias 0.02 deg/sec.
  - Bandwidth 25 Hz

- **Acceleration:**
  - Scale factor 1%
  - Bias 5 mg
  - Bandwidth 20 Hz
Accuracy

• **Commercial cruise systems**
  – Position: 0.2 nautical miles of error per hour of operation.
    • In some cases, position accuracy along the trajectory (alongtrack) and both normal directions (crosstrack and vertical) are distinguished.
  – Attitude (pitch and roll): often accurate to 0.05°.
  – Heading: often accurate to 0.5°.

• **Land vehicle navigation systems:**
  – Position: 0.2% to 2% of distance traveled.
  – Attitude: 0.1°
  – Heading to 0.5°.
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6.3.4 Simple Odometry Aided AHRS

- The AHRS is a degenerate form of inertial navigation system, using much of the same components:
  - indicates orientation only.
- Device uses a strapped down IMU today.
  - Accels indicate gravity and acceleration
  - Gyros indicate angular velocity
- Distinguishing acceleration from gravity is still an issue - but less so.
6.4.3.1 Nav Eqns in Body Frame

• Recall the inertial nav equation (Eq 6.46):
  \[
  \dot{\vec{a}}_v = \left( \frac{d\vec{v}_v}{dt} \right)_e = \ddot{t} - 2\vec{\Omega} \times \vec{v}_v + \vec{g}
  \]

• Let's express this in the body frame so that it becomes unnecessary to known orientation.

• Use the Coriolis theorem:

  \[
  \left( \frac{d\vec{v}_v}{dt} \right)_v = \left( \frac{d\vec{v}_v}{dt} \right)_e + \vec{\Omega}_v \times \vec{v}_v
  \]

  This adds another Apparent Coriolis Force.
6.4.3.1 Nav Eqns in Body Frame

- Define the strapdown angular velocity:

\[
\dot{\omega} = \omega_v = \omega_v + \omega_e = \omega_v + \Omega
\]

- Write the inertial navigation equation in the body frame:

\[
\left( \frac{dv_v^e}{dt} \right)_v = \dot{t} - (\dot{\omega} + \Omega) \times v_v^e + \dot{g}
\]

- For this purpose, earth rate can be neglected, so:

\[
\left( \frac{dv_v^e}{dt} \right)_v = \dot{t} - \dot{\omega} \times v_v^e + \dot{g}
\]

Recall Earth Frame
6.4.3.1 Nav Eqns in Body Frame

• Solve for gravity:

\[ \hat{g} = \begin{pmatrix} \frac{dv^e_v}{dt} \\ - \imath + \omega \times v^e_v \end{pmatrix} \]

This vanishes on an Ackerman vehicle during periods of constant speed. Otherwise, differentiate numerically.

Simply remove Coriolis term from the accel readings.

• Everything on right is known from measurements. g on left is known in world coordinates.
6.4.3.1 Nav Eqns in Body Frame

• Write this in body coordinates:

\[ R^v_{w} = \frac{d(v^e_v)}{dt} - t + \omega \times v^e_v = v^g_{meas} \]

• Can solve this for attitude (not yaw) in the rotation matrix using inverse kinematics.
  
  − Rotation around g is not observable.
6.4.3.2 Solving for Attitude

• To get the attitude, express in body frame:

\[ R_w^y g = \frac{d}{dt} v_v^e - t + \omega \times v_v^e = g_{meas} \]

• Where:

\[ Roty(\theta)Rotx(\phi)g = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \begin{bmatrix} 0 \\ g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \]

• The transpose converts from world to body, thus:

\[ g_{meas}^v = R_w^y g = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ s\theta s\phi & c\phi & c\theta s\phi \\ s\theta c\phi & -s\phi & c\theta c\phi \end{bmatrix} \begin{bmatrix} 0 \\ -s\theta \\ c\theta s\phi \end{bmatrix} = g \begin{bmatrix} -s\theta \\ c\theta s\phi \\ c\theta c\phi \end{bmatrix} \]
6.4.3.2 Solving for Attitude

• The solution is:

\[ \tan \theta = \frac{s \theta}{c \theta} = -\frac{g_x}{\sqrt{g_y^2 + g_z^2}} \]
\[ \tan \phi = \frac{s \phi}{c \phi} = \frac{g_y}{g_z} \]

• To get the yaw rate, solve:

\[ \dot{\psi} = \frac{s \phi}{c \theta} \omega_y + \frac{c \phi}{c \theta} \omega_z \]
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Summary

• Black magic?
• Hard to do well.
  – Costs big bucks.
• Most accurate dead reckoning available.
  – Cruise: 0.2 nautical miles of error per hour of operation.
• Indispensable on outdoor mobile robots.
• Complementary technology to GPS.
Summary

• Inertial navigation is based on Newton’s laws
  – Works everywhere that gravity is known.
  – It is stealthy and jamproof.

• Modern “strapdown” systems
  – “computationally stabilized”.
  – no stabilized platform

• Naive approaches are seriously flawed. Must compensate for
  – Gravity
  – inertial forces
  – body fixed coordinates.
Summary

• Free inertial performs miserably...
  – 1 part in 10,000 acceleration error causes kilometers of position error after 1 hour of operation.

• Interesting Error Dynamics
  – Horizontal errors bounded, oscillate every 84 minutes
  – Vertical position is unstable without damping devices

• An AHRS unit can find attitude from accelerometers and gyros and odometry.