Chapter 4
Dynamics

Part 1
4.1 Moving Coordinate Systems
4.2 Kinematics of WMRs
Outline

• 4.1 Moving Coordinate Systems
  – 4.1.1 Context of Measurement
  – 4.1.2 Change of Reference Frame
  – 4.1.3 Example: Attitude Stability Margin
  – 4.1.4 Recursive Transformation of States of Motion
  – Summary

• 4.2 Kinematics of Wheeled Mobile Robots
Outline

• 4.1 Moving Coordinate Systems
  – 4.1.1 Context of Measurement
  – 4.1.2 Change of Reference Frame
  – 4.1.3 Example: Attitude Stability Margin
  – 4.1.4 Recursive Transformation of States of Motion
  – Summary

• 4.2 Kinematics of Wheeled Mobile Robots
4.1.1 Context of Measurement

• Police says “you were going 30 m/s southbound on I-279”.

• Any measurement in physics lacks meaning without several contextual elements:
  – a unit system (e.g. meters, seconds)
  – a number system (e.g. base 10 weighted positional)
  – a coordinate system (e.g. directions north, east)
  – a reference frame to which the measurement is ascribed (e.g. your car).
  – a reference frame with respect to which the measurement is made (e.g. the earth).
Coordinate Systems

• **Conventions for representation** of physical quantities.
  – Any set of quantities that fixes all degrees of freedom of a system.
  – Cartesian systems represent vectors by projections onto three orthogonal axes.
  – The Euler angle definition expresses the three degrees of rotational freedom.

• **Mathematical laws alone** govern conversion from coordinate system to coordinate system.

• Conversion of coordinates **does not change** the magnitude or direction of a measurement - only the way you describe it.
Reference Frames

• Allows us to reconcile the differences in observations of the same property of an object by two observers with different states of motion.
  – Laws of physics are necessary to convert among frames of reference (i.e. to predict a measurement made by one observer from those of another).
  – A reference frame is a real physical body. The state of motion of such a body distinguishes it from other frames of reference.

• A phenomenon, when observed from one frame of reference, may or may not look the same when observed from a second frame of reference.

• Two frames are equivalent with respect to a measurement when the measurement is the same in both frames. If they are not equivalent, a method of converting between the frames of reference is often available.
Outline

• 4.1 Moving Coordinate Systems
  – 4.1.1 Context of Measurement
  – 4.1.2 Change of Reference Frame
  – 4.1.3 Example: Attitude Stability Margin
  – 4.1.4 Recursive Transformation of States of Motion
  – Summary

• 4.2 Kinematics of Wheeled Mobile Robots
Coriolis

• Cauchy recommended him to a job at Ecole Polytechnique.
• Introduced the terms 'work' and 'kinetic energy' with their present scientific meaning
• Best remembered for “Sur les équations du mouvement relatif des systèmes de corps (1835)” which introduced the Coriolis force.
• Also wrote a mathematical theory of billiards!

Gaspard-Gustave de Coriolis
1792-1843 Paris, France
4.1.2.1 Mutually Stationary Frames

- Motion of particle can be expressed wrt either frame.
- Position vectors are related:
  \[ \overrightarrow{r_p} = \overrightarrow{r_p} + \overrightarrow{r_h} \]
- Differentiate wrt time:
  \[ \overrightarrow{a} = \overrightarrow{v_p} \]
4.1.2.2 Galilean Transformation
Translating (Const V) Frames

• Motion of particle can be expressed wrt either frame.

• Position vectors are related:

\[ \vec{r}_p^t = \vec{r}_p^t + \vec{r}_{f0}^t + \vec{v}_f \cdot t \]

• Differentiate wrt time:

\[ \vec{v}_p^t = \vec{v}_p^t + \vec{v}_f \]

• Frames are equivalent for acceleration:

\[ \vec{a}_p^t = \vec{a}_p^t \]

Relates particle velocity for an observer in the control tower to that observed in the airplane.

Called “Galilean Transformation”
4.1.2.3 Rotating Frames

• When two frames are rotating with respect to each other, something must be accelerating.
• Let $\omega$ denote angular velocity of m frame with respect to f frame.
• Let's predict measurements of observer in f given those of observer in m.
4.1.2.3 Coriolis Equation

• Coriolis Equation (aka Transport Theorem) relates derivatives of same vector by both observers.

\[
\left( \frac{du}{dt} \right)_f = \left( \frac{du}{dt} \right)_m + \vec{\omega} \times \vec{u}
\]

• \( \vec{u} \) is any vector (position, velocity, acceleration, force)

• \( \vec{\omega} \) is angular velocity of moving frame wrt fixed one.
4.1.2.4 Velocity Transformation

- Positions add by vector addition.
  \[ \dot{\mathbf{r}}_o^f = \dot{\mathbf{r}}_m^f + \dot{\mathbf{r}}_o^m \]

- Time derivative in fixed frame.
  \[ \frac{d}{dt} \bigg|_{\mathbf{r}_o^f} = \frac{d}{dt} \bigg|_{\mathbf{r}_m^f} + \frac{d}{dt} \bigg|_{\mathbf{r}_o^m} \]

- Apply Coriolis Equation to 2\textsuperscript{nd} term on right.
  \[ \dot{\mathbf{v}}_o^f = \dot{\mathbf{v}}_m^f + \omega_m^f \times \dot{\mathbf{r}}_o^m + \dot{\mathbf{v}}_o^m \]

Note:

\[ \frac{d}{dt} \bigg|_{\mathbf{r}_y^x} = \dot{\mathbf{v}}_y^x \]
4.1.2.4 General Velocity Relation

\[ \vec{v}_o = \vec{v}_m + \vec{\omega}_m \times \vec{r}_o + \vec{v}_o \]

\( \vec{v}_o \): velocity of particle relative to fixed observer
\( \vec{v}_m \): velocity of particle relative to moving observer
\( \vec{v}_m \): linear velocity of moving observer relative to fixed
\( \vec{\omega}_m \): angular velocity of moving observer relative to fixed
\( \vec{r}_o \): position of particle relative to moving observer
4.1.2.5 General Acceleration Relation

- Apply to velocity relation

\[
\frac{d}{dt} (\mathbf{\dot{v}}_o) = \frac{d}{dt} \left( \mathbf{\dot{v}}_m + \mathbf{\alpha}_m \times \mathbf{r}_o + \mathbf{\dot{r}}_o \right)
\]

\[
\frac{d}{dt} (\mathbf{\dot{v}}_o) = \frac{d}{dt} (\mathbf{\dot{v}}_m) + \frac{d}{dt} (\mathbf{\alpha}_m \times \mathbf{r}_o) + \frac{d}{dt} (\mathbf{\dot{r}}_o)
\]

\[
\mathbf{\ddot{f}}_o = \mathbf{\ddot{f}}_m + \mathbf{\alpha}_m \times \mathbf{\dot{r}}_o + \mathbf{\omega}_m \times [ \mathbf{\alpha}_m \times \mathbf{r}_o ] + 2 \mathbf{\omega}_m \times \mathbf{\dot{v}}_o + \mathbf{\ddot{r}}_o
\]
4.1.2.5 General Acceleration Relation

\[
\ddot{a}_o = \ddot{a}_m + \alpha_m \times \dot{r}_o + \omega_m \times [\omega_m \times \dot{r}_o] + 2\omega_m \times \dot{v}_o + \ddot{a}_o
\]

\[\ddot{a}_m\] : Einstein acceleration (of moving frame wrt fixed)

\[\ddot{a}_o\] : acceleration of particle relative to fixed observer

\[\ddot{a}_m\] : acceleration of particle relative to moving observer

\[2\omega_m \times \dot{v}_o\] : Coriolis acceleration

\[\alpha_m \times \dot{r}_o\] : Euler acceleration

\[\omega_m \times [\omega_m \times \dot{r}_o]\] : Centripetal acceleration
Outline

• 4.1 Moving Coordinate Systems
  – 4.1.1 Context of Measurement
  – 4.1.2 Change of Reference Frame
  – 4.1.3 Example: Attitude Stability Margin
  – 4.1.4 Recursive Transformation of States of Motion
  – Summary

• 4.2 Kinematics of Wheeled Mobile Robots
4.1.3 Attitude Stability Margin Estimation

- Staying upright
- Keeping contact with terrain.
- Important when:
  - Lifting heavy loads
  - Turning at speed
  - Operating on sloped terrain
- Many vehicles do one or more of these things.
Liftoff Criterion

• Preventing liftoff will prevent rollover.

• For liftoff, issue is the direction of the net noncontact force vector acting at the cg
  – Any unbalanced moment about any tipover axis.
4.1.3.1 Proximity to Wheel Liftoff

- Place a 2 axis accel right at the cg.

- BUT:
  - CG may not be accessible.
  - It may move due to:
    - articulations
    - payload changes
    - changing human passengers.
4.1.3.1 Proximity to Wheel Liftoff

• Define the specific force acting at the cg:

\[ \hat{t} = \hat{a} - \hat{g} \]

• An accelerometer can measure specific force but it cannot usually be placed at the cg.
  – Therefore transform it.

Vehicle is viewed from rear when executing a hard left turn. For the specific force direction shown, the inner (left) wheels are about to lift off.
4.1.3.1 Proximity to Wheel Liftoff

Transformation

- Use earlier result:
  - $f$ frame in inertial (i) frame.
  - $m$ frame is sensor (s) frame.
  - $o$ frame is cg (c) frame

- Simply substitute the letters to get:

$$\dot{a}_c = \dot{a}_s + \dot{a}_s \times r_c + \omega_s \times [\omega_s \times r_c] + 2\omega_s \times v_c + a_c$$

- Subtract the gravity vector from both sides to get the real (s) and transformed (c) accelerometer readings
4.1.3.3 Computational Requirements

• Geometry
  – Location of the center of gravity (cg).
  – Convex polygon formed by the wheel contact points.

• Forces
  – Gravity vector.
  – Inertial forces being experienced due to accelerated motion.
Outline

• 4.1 Moving Coordinate Systems
  – 4.1.1 Context of Measurement
  – 4.1.2 Change of Reference Frame
  – 4.1.3 Example: Attitude Stability Margin
  – 4.1.4 Recursive Transformation of States of Motion
  – Summary

• 4.2 Kinematics of Wheeled Mobile Robots
4.1.4 Recursive Transformation of States of Motion

- Suppose we have a sequence of frames numbered:
  - 1, 2, ..., k, k+1, ..., n
- Their motions can be related by the results just derived...

\[
\begin{align*}
\dot{r}_o^k &= \dot{r}_{k+1}^k + \dot{r}_o^{k+1} \\
\dot{v}_o^k &= \dot{v}_{k+1}^k + \omega_{k+1}^k \times \dot{r}_o^k + \dot{v}_o^{k+1} \\
\dot{a}_o^k &= \ddot{a}_{k+1}^k + \gamma_{k+1}^k \times \dot{r}_o^k + \ddot{a}_{k+1}^k \times \left[ \omega_{k+1}^k \times \dot{r}_o^k \right] + 2 \omega_{k+1}^k \times \ddot{v}_o^k + \dot{a}_o^{k+1}
\end{align*}
\]
4.1.4.1 Conversion to Coordinatized Form

- Recall:
  \[ \vec{\omega} \times \vec{u} \Rightarrow \vec{\omega} \times \dot{u} = [\vec{\omega}]^\times \dot{u} = -[\dot{u}]^\times \vec{\omega} \]

- Where:
  \[ [\dot{u}]^\times \equiv \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \]

- Also, we can write the transport theorem in matrix form:
  \[ \left( \frac{du}{dt} \right)_f = \left( \frac{du}{dt} \right)_m + [\vec{\omega}]^\times \dot{u} \]
4.1.4.1 Conversion to Coordinatized Form

• Now, use this to rewrite Equation 4.18.

\[
\begin{align*}
\rho^k_{-o} &= \rho^k_{-k+1} + \rho^{k+1}_{-o} \\
\nu^k_o &= \begin{bmatrix} I & -[\rho^k_{-o}] \end{bmatrix} \times \begin{bmatrix} v^k_{k+1} \\
\omega^k_{k+1} \end{bmatrix} + \nu^k_{o} \\
a^k_{-o} &= \begin{bmatrix} I & -[\rho^k_{-o}] \end{bmatrix} \times \begin{bmatrix} a^k_{k+1} \\
\omega^k_{k+1} \end{bmatrix} + \begin{bmatrix} \omega^k_{k+1} \times \times \times \times 2[\omega^k_{k+1}] \times \times \times \times I \\
\nu^k_{o} \\
a^k_{-o} \end{bmatrix}
\end{align*}
\]
4.1.4.1 Conversion to Coordinatized Form

• Define the notation:

\[ \mathbf{p} \equiv \begin{bmatrix} r ; v ; a \end{bmatrix} \]

\[ \mathbf{x} \equiv \begin{bmatrix} r ; v ; \omega ; a ; \alpha \end{bmatrix} \]

• Then, previous position, velocity and acceleration results can be written as:

\[ \mathbf{p}^k_o = H(\mathbf{p}^{k+1}_o)\mathbf{x}^k + \Omega(\mathbf{x}^{k+1}_o)\mathbf{p}^{k+1}_o \]

• Typically

- \[ \mathbf{x}^k \]
- \[ \mathbf{p}^k_o \]

represent articulations

represent state of motion of each frame
4.1.4.2 General Recursive Forms

Velocity Transform

• Consider just the velocity transform:

\[
\begin{bmatrix}
    v_{o}^k \\
    \omega_{o}^k
\end{bmatrix} = \begin{bmatrix}
    I & -[L_{o}^{k+1}]
\end{bmatrix} \begin{bmatrix}
    v_{k+1}^k \\
    \omega_{k+1}^k
\end{bmatrix} + \begin{bmatrix}
    v_{o}^{k+1} \\
    \omega_{o}^{k+1}
\end{bmatrix} \tag{4.23}
\]

• We will write this compactly as:

\[
\begin{bmatrix}
    v_{o}^k \\
    \omega_{o}^k
\end{bmatrix} = H(r_{o}^{k+1})\begin{bmatrix}
    x_{o}^k \\
    \omega_{o}^{k+1}
\end{bmatrix} + \begin{bmatrix}
    v_{o}^{k+1} \\
    \omega_{o}^{k+1}
\end{bmatrix} \tag{4.24}
\]

• Where H is defined as it occurs in Equation 4.23.
4.1.4.2 General Recursive Forms

Acceleration Transform

• Consider just the acceleration transform:

\[
\begin{bmatrix}
\frac{a^k}{-o} \\
\frac{a^{k+1}}{k+1}
\end{bmatrix}
= \begin{bmatrix}
I \\
-\alpha^k + 1
\end{bmatrix}
\begin{bmatrix}
\frac{a^k}{k} \\
\frac{a^{k+1}}{k+1}
\end{bmatrix}
+ \begin{bmatrix}
\Omega^k + 1 \\
2\Omega^k + 1
\end{bmatrix}
\begin{bmatrix}
\frac{r^k}{-o} \\
\frac{v^k}{k+1}
\end{bmatrix}
+ \frac{a^{k+1}}{k+1}
\]

(4.25)

• We will write this compactly as:

\[
\frac{a^k}{-o} = H(r^k + 1) + \Omega(\Omega^k + 1) + \frac{a^{k+1}}{k+1}
\]

(4.26)

• Where \( H, \Omega \) etc. are defined as they occur in Equation 4.25.
4.1.4.3 The Articulated Wheel

• Let \( n = k + 1 \) have a maximum value of 2.
  – Two intermediate frames relate zeroth frame (0) to object frame (o).

• Write Equation 4.24 twice:

\[
\begin{align*}
\dot{y}_o^0 &= H(r_o^1)\dot{x}_1^0 + v_o^1 \\
\dot{y}_o^1 &= H(r_o^2)\dot{x}_2^1 + v_o^2
\end{align*}
\]

• Substitute second into first:

\[
\dot{y}_o^0 = H(r_o^1)\dot{x}_1^0 + H(r_o^2)\dot{x}_2^1 + v_o^2 \quad (4.27)
\]
4.1.4.3 The Articulated Wheel

• For acceleration, write Equation 4.26 twice:

\[
\ddot{a}_o = H(r_o^1)\ddot{x}_1^0 + \Omega(\omega_1^0)\dot{p}_o^1 + \ddot{a}_o
\]

\[
\ddot{a}_o = H(r_o^2)\ddot{x}_2^1 + \Omega(\omega_2^1)\dot{p}_o^2 + \ddot{a}_o
\]

• Substitute second into first:

\[
\ddot{a}_o = H(r_o^1)\ddot{x}_1^0 + \Omega(\omega_1^0)\dot{p}_o^1 + H(r_o^2)\ddot{x}_2^1 + \Omega(\omega_2^1)\dot{p}_o^2 + \ddot{a}_o
\]

(4.29)
4.1.4.4 Velocity Transforms for Articulated Wheel

Let these frames be defined:
- 0: world frame (w)
- 1: body frame (v)
- 2: suspension/steering (s)
- o: wheel contact pt (c)

Then equation 4.27 becomes:

\[
\begin{align*}
\dot{v}_c^w &= H(r_c^v)x_v^w + H(r_c^s)x_s^v + v_c^s \\
\dot{v}_c^w &= \begin{bmatrix} I & -[r_c^v]_x \end{bmatrix} \begin{bmatrix} v_v^w \\ \omega_v^w \end{bmatrix} + \begin{bmatrix} I & -[r_c^s]_x \end{bmatrix} \begin{bmatrix} v_s^v \\ \omega_s^v \end{bmatrix} + v_c^s
\end{align*}
\]

(4.30)
4.1.4.4 Velocity Transforms for Articulated Wheel

- Under the same substitutions
  Equation 4.29 becomes:

\[
\begin{align*}
\dot{a}^w_c &= H(\dot{r}_c^v) \dot{x}^w_v + \Omega(\omega_v^w) \dot{p}_c^v + H(\dot{r}_c^s) \dot{x}^s_s + \Omega(\omega_s^v) \dot{p}_c^s + \ddot{a}_c^s \\
\dot{a}^w_c &= \begin{bmatrix} I & -[r_c^v] \end{bmatrix} \begin{bmatrix} \dot{a}_v^w \\ \Omega \omega_v^w \end{bmatrix} + \begin{bmatrix} \dot{\omega}_v^w \times \omega_v^w \\ 2\dot{\omega}_v^w \times \omega_v^w \end{bmatrix} \begin{bmatrix} r_c^v \\ v_c^v \end{bmatrix} \\
&\quad + \begin{bmatrix} I & -[r_c^s] \end{bmatrix} \begin{bmatrix} \dot{a}_s^v \\ \Omega \omega_s^v \end{bmatrix} + \begin{bmatrix} \dot{\omega}_s^v \times \omega_s^v \\ 2\dot{\omega}_s^v \times \omega_s^v \end{bmatrix} \begin{bmatrix} r_c^s \\ v_c^s \end{bmatrix} + \ddot{a}_c^s
\end{align*}
\]

(4.31)

Articulated Wheel Acceleration Kinematics
Outline

• 4.1 Moving Coordinate Systems
  – 4.1.1 Context of Measurement
  – 4.1.2 Change of Reference Frame
  – 4.1.3 Example: Attitude Stability Margin
  – 4.1.4 Recursive Transformation of States of Motion
  – Summary

• 4.2 Kinematics of Wheeled Mobile Robots
Summary

• Measurements require a context to be precisely meaningful.
• A coordinate system and a reference frame are different.
• Two frames may or may not be equivalent for measuring velocity and higher derivatives.
• The Coriolis Equation provides a general coordinate-free mechanism to differentiate any vector attached to a moving frame of reference.
  – General transformations of position, velocity, and acceleration can be derived from it.
• Basic stability margin estimation is based on lift-off and an acceleration transformation.
• A two step recursion is sufficient to model the velocity and acceleration kinematics relating the wheel contact point motion to the motion and articulation of a WMR.
Outline

• 4.1 Moving Coordinate Systems
• 4.2 Kinematics of Wheeled Mobile Robots
  – 4.2.1 Aspects of Rigid Body Motion
  – 4.2.2 WMR Velocity Kinematics for Fixed Contact Point
  – 4.2.3 Common Steering Configurations
  – Summary
4.2.1.1 Pure Rotation of a Point

• Suppose particle p moves in pure rotation.

\[ \dot{r}_p = r [\cos(\psi)i + \sin(\psi)j] \]

• Differentiate:

\[ \dot{v}_p = r \omega [-\sin(\psi)i + \cos(\psi)j] \]

• ... orthogonal to \( \dot{r}_p \)

• In other “words”

\[ \dot{v}_p = \omega \times \dot{r}_p \]

• Magnitudes are:

\[ v_p = r_p \omega \]  

Eqn A
Pure Rotation of a Particle on a Body

• Now consider a particle \( p \) on a body executing general planar motion.
  – Not pure rotation...

• The body has some some \( V \) and \( \omega \) at the position of \( p \).

• For some world frame \( W \), define the ratio.
  \[
  r = \frac{v_p^W}{\omega}
  \]
Pure Rotation of a Particle on a Body

- Rewrite this as:
  \[ r = \frac{v_p^w}{\omega} \]

- This is Eqn A! Hence we can interpret ‘r’ as the radius to an instantaneous center of rotation (ICR) for point p located r units in orthogonal direction to v.

- In vector terms:
  \[ \vec{v}_{icr}^p = \vec{\omega} \times \vec{r}_{icr}^p \]
Pure Rotation of a Particle on a Body

• Consider a neighboring point \( q \):

\[
\dot{r}_{q}^{icr} = \dot{r}_{p}^{icr} + \dot{r}_{q}^{p}
\]

• Differentiate in the world frame:

\[
\frac{d}{dt}\bigg|_{w} (\dot{r}_{q}^{icr}) = \frac{d}{dt}\bigg|_{w} (\dot{r}_{p}^{icr}) + \frac{d}{dt}\bigg|_{w} (\dot{r}_{q}^{p})
\]

• But the last term is:

\[
\frac{d}{dt}\bigg|_{w} (\dot{r}_{q}^{p}) = \frac{d}{dt}\bigg|_{b} (\dot{r}_{q}^{p}) + \dot{\omega} \times \dot{r}_{q}^{p} = \dot{\omega} \times \dot{r}_{q}^{p}
\]
Pure Rotation of a Particle on a Body

Substituting:

\[
\vec{v}_{q} = \vec{v}_{p} + \vec{\omega} \times \vec{r}_{q}
\]

Substitute for the first term:

\[
\vec{v}_{q} = \vec{\omega} \times \vec{r}_{p} + \vec{\omega} \times \vec{r}_{q} = \vec{\omega} \times (\vec{r}_{p} + \vec{r}_{q})
\]

That is:

\[
\vec{v}_{q} = \vec{\omega} \times \vec{r}_{q}
\]

Every point on the body is executing a pure rotation about the ICR.
4.2.1.2 Jeantaud Diagrams

- Fixing just the directions of the velocities of two points on a body determines the ICR.
- Hence, all steered wheels of a vehicle must be consistent to avoid wheel slip and energy loss.
- Wheels do not slip if they move along the normal to the line to the ICR.
- If all wheels are consistent, any two directions and one velocity can be used to predict the motion.

This is called a Jeantaud Diagram.
This vehicle can "crab steer".
4.2.1.3 Rolling Contact

• Wheels normally have up to two degrees of freedom.
  – steer
  – drive

• Angular and linear velocity are related as follows ...

\[ v_c = r \omega \]

  – under a no slip assumption:
4.2.1.4 Rolling without Slipping

• This constraint means $\dot{x}$ and $\dot{y}$ are not independent.
  – They must be aligned with the direction of pure rolling.

• The dot product ...
  – $[\dot{x} \quad \dot{y}] \cdot [s \psi \quad -c \psi] = 0$

• Written out ...
  – $\dot{x} s \psi - \dot{y} c \psi = 0$

Disallowed Direction
Pfaffian Constraints

• Define the wheel configuration vector:

\[ x = \begin{bmatrix} x & y & \psi & \theta \end{bmatrix}^T \]

• And the weight vector:

\[ w(x) = \begin{bmatrix} \sin \psi & -\cos \psi & 0 & 0 \end{bmatrix} \]

• The constraint in Pfaffian form is:

\[ w(x)\dot{x} = 0 \]
Nonholonomic Constraints

• Typically (not always) wheels cannot move sideways (without slipping).
• Creates severe mathematical difficulties.
• Most wheels, and therefore most WMR’s, are subject to these nonholonomic constraints.
Definition

• Such constraints are “nonholonomic” because they cannot be put in the form:

\[ c(x) = 0 \]

• The integral would be:

\[
\int_{0}^{t} w(x) \dot{x} \, dt = \int_{0}^{t} (\dot{x} \sin \psi(t) - \dot{y} \cos \psi(t)) \, dt
\]

• Even when \( \psi(t) = t^2 \), these integrals are the well-known Fresnel integrals which have no closed form solution.

  – And hence cannot be reduced to the form \( c(x) = 0 \).
Outline

• 4.1 Moving Coordinate Systems

• 4.2 Kinematics of Wheeled Mobile Robots
  – 4.2.1 Aspects of Rigid Body Motion
  – 4.2.2 WMR Velocity Kinematics for Fixed Contact Point
  – 4.2.3 Common Steering Configurations
  – Summary
4.2.2 Character of WMR Models

• Unlike manipulators, the simplest models of how mobile robots move are differential equations that are:
  – Nonlinear
  – Underactuated
  – Constrained

\[ \dot{x} = f(x, u) \]

\[ w(x) \dot{\dot{x}} = 0 \]

• Much of the difficulty of mobile robots can be traced to this fact.
Motion Prediction

• The process of integrating the differential equations for known inputs can be called motion prediction. It is important for:
  • estimating state in odometry, Kalman filter system models, and more generally in pose estimation of any kind.
  • predicting state in predictive control
  • simulating motion in simulators.
Rate Kinematics

• For (WMRs), we care about the rate kinematics.
• Of basic interest are two questions:

  – For state estimation

  – For control
Frame Conventions

- **w**: world
- **v**: vehicle
- **s**: steer
- **c**: contact point.
- Regard vehicles as rigid bodies (no suspension).
  - Except for steering and wheel rotation.
- Contact point moves on wheel and on floor but it is fixed in wheel frame.
Offset Wheel Equation

• Key assumption: wheel contact point is fixed to wheel. So...
  \[ \dot{v}_s^w = \dot{v}_c^w = 0 \]

• Eqn 4.30 becomes
  \[
  \dot{v}_c^w = \dot{v}_v^w - [r_{c}^v] \times \omega_v^w - [r_{c}^s] \times \omega_s^v
  \] (4.39)

• When s and c frames are coincident
  \[
  \dot{v}_c^w = \dot{v}_v^w - [r_{c}^v] \times \omega_v^w
  \] (4.40)
4.2.2.1.1 Wheel Steering Control

• For steering, note that **direction** (not magnitude) of s frame and c frame velocities must be parallel.

• So, propagate velocity from v frame to s frame:

\[ \dot{v}_s^w = \dot{v}_v^w - [r_s^v] \times \omega_v^w \]

• Express in vehicle coordinates and extract steer angle:

\[ \gamma = \text{atan2}[(\dot{v}_s^w)_y, (\dot{v}_s^w)_x] \]
4.2.2.1.2 Wheel Speed Control

• Assuming
  – a) the wheels are steered appropriately
  – b) no slip

• Then, the magnitude of wheel speed is also the component in the forward direction.

• Compute it in vehicle coordinates where posn vectors are easy to get:

\[ v^w_c = v^w_v - [r^v_{c}] \times v^w_\omega v - [R^v_s r^s_{c}] \times v^v_\omega s \]

• That gives the wheel speed as

\[ v^w_c = \sqrt{(v^w_{c,x})^2 + (v^w_{c,y})^2} \quad \omega_{\text{wheel}} = \frac{v^w_c}{r_{\text{wheel}}} \]
4.2.2.2.1 Wheel Sensing

• Wheel linear speed:

\[ v_k = r_k \omega_k \]

• Wheel speed components:

\[ (v_k)_x = v_k \cos(\gamma_k) \]
\[ (v_k)_y = v_k \sin(\gamma_k) \]
4.2.2.2.2 Multiple Offset Wheels

• Write the offset wheel equation in vehicle coordinates:

\[ \mathbf{\nu}_c = \mathbf{\nu}_v - [r^v_c] \times \mathbf{\omega}_v \mathbf{w} - [R^v_s r^s_c] \times \mathbf{\omega}_s \mathbf{w} \]

• This is of the form:

\[ \mathbf{\nu}_c = H^v_c(\gamma) \begin{bmatrix} \mathbf{\nu}_v \\ \mathbf{\omega}_v \end{bmatrix} + Q^s_c(\gamma) \mathbf{\omega}_s \mathbf{w} \]

• If we write one of these for each wheel, stack em up, the result looks like:

\[ \mathbf{\nu}_c = H^v_c(\gamma) \dot{\mathbf{x}}_v + Q^s_c(\gamma) \cdot \gamma \]
4.2.2.2.2 Multiple Offset Wheels (Inv)

- Result from last slide again is:
  \[
  \dot{\mathbf{v}}^w_c = H^v_c(\gamma) \dot{\mathbf{x}}^w_v + Q^s_c(\gamma) \dot{\gamma}
  \]

- The LHS and steer angles are known, and this is normally overdetermined, so use the left pseudoinverse:

\[
\dot{\mathbf{x}}^w_v = \left[ H^v_c(\gamma)^T H^v_c(\gamma) \right]^{-1} H^v_c(\gamma)^T \left[ \mathbf{v}^w_c - Q^s_c(\gamma) \dot{\gamma} \right]
\]

- Robot linear and angular velocity
- Steer Angles
- Wheel Speeds
- Steer Angle Rates
Box 4.2: WMR Forward Kinematics: Offset Wheels

Offset wheel equations for all wheels can be grouped together to produce

\[ \gamma^w_c = H_c^v(\gamma) \dot{x}_v^w + Q_c^s(\gamma) \dot{\gamma} \]

where each pair of rows of \( H_c^v \) and \( Q_c^s \) comes from an offset equation expressed in body coordinates, \( \gamma^w_c \) is the wheel velocities, \( \dot{x}_v^w \) is the linear and angular velocity of the vehicle, and \( \dot{\gamma} \) is the steer angles.

The inverse mapping (for two or more wheels) can be computed with:

\[ \dot{x}_v^w = \left[ H_c^v(\gamma)^T H_c^v(\gamma) \right]^{-1} H_c^v(\gamma)^T \left[ \gamma^w_c - Q_c^s(\gamma) \dot{\gamma} \right] \]

For nonoffset wheels \( H_c^v \) simplifies, and \( Q_c^s \) disappears.
Outline

• 4.1 Moving Coordinate Systems
• 4.2 Kinematics of Wheeled Mobile Robots
  – 4.2.1 Aspects of Rigid Body Motion
  – 4.2.2 WMR Velocity Kinematics for Fixed Contact Point
  – 4.2.3 Common Steering Configurations
  – Summary
Example: Differential Steer (Inv)

- Let ‘l’ and ‘r’ denote left and right wheel frames.
- The dimensions are:
  \[
  \ell_l^v = \begin{bmatrix} 0 & W \end{bmatrix}^T, \quad \ell_r^v = \begin{bmatrix} 0 & -W \end{bmatrix}^T
  \]
- In body frame, velocities have only an x component. Equation 4.40 reduces to:
  \[
  \begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} v_x + \omega W \\ v_x - \omega W \end{bmatrix} = \begin{bmatrix} 1 & W \\ 1 & -W \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}
  \]
- Two equations giving sideways wheel velocities were of the form \( v_y = 0 \), so these were not written.

Can solve for 2 dof of 3 dof motion. Other dof is zero in body frame (for this choice of body frame).
Example: Differential Steer (Fwd)

• Inverse kinematics again are:

\[
\begin{bmatrix}
  v_r \\
  v_l
\end{bmatrix} = \begin{bmatrix}
  1 & W \\
  1 & -W
\end{bmatrix} \begin{bmatrix}
  v_x \\
  \omega
\end{bmatrix}
\]

• This is easy to invert:

\[
\begin{bmatrix}
  v_x \\
  \omega
\end{bmatrix} = \frac{1}{2W} \begin{bmatrix}
  W & W \\
  1 & -1
\end{bmatrix} \begin{bmatrix}
  v_r \\
  v_l
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
  1 & 1 \\
  \frac{1}{W} & -\frac{1}{W}
\end{bmatrix} \begin{bmatrix}
  v_r \\
  v_l
\end{bmatrix}
\]

Again, other dof is zero in body frame due to nonholonomic constraints.
Example: Ackerman Steer

• Special mechanism ensures wheels are lined up properly.
Example: Ackerman Steer (Inverse)

- Position vector to front wheel in body (vehicle) frame:
  \[
  \ell_f^v = \begin{bmatrix} L \\ 0 \end{bmatrix}^T
  \]

- Cross product skew matrix:
  \[
  [\ell_f^v]^\times = \begin{bmatrix}
  0 & -(\ell_f^v)_z & (\ell_f^v)_y \\
  (\ell_f^v)_z & 0 & -(\ell_f^v)_x \\
  -(\ell_f^v)_y & (\ell_f^v)_x & 0
  \end{bmatrix}
  = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & -L \\
  0 & L & 0
  \end{bmatrix}
  \]

- Wheel equation in body frame reduces to:
  \[
  v_f^w = v_v^w - [\ell_c^v]^\times \omega_v^w \Rightarrow v_f = \begin{bmatrix} v_x \\ \omega L \end{bmatrix}^T
  \]

BTW rear wheel velocity is trivial
Example: Ackerman Steer (Inverse)

- Last result is of the form:
  \[ \mathbf{v}_f = H_c \mathbf{x}_v \]

- Written out:
  \[
  \begin{bmatrix}
  v_{fx} \\
  v_{fy}
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 & 0 \\
  0 & L
  \end{bmatrix}
  \begin{bmatrix}
  v_x \\
  \omega
  \end{bmatrix}
  \]

- So, the angle of the front wheel can be computed:
  \[
  \tan(\gamma) = \frac{\omega L}{v_x} = \kappa L = \frac{L}{R}
  \]

The "Car" Equation
Example: Ackerman Steer (Fwd)

- Inverse kinematics again are:

\[
\begin{bmatrix}
  v_{fx} \\
  v_{fy}
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 \\
  0 & L
\end{bmatrix}
\begin{bmatrix}
  v_x \\
  \omega
\end{bmatrix}
\]

- This is easy to invert:

\[
\begin{bmatrix}
  v_x \\
  \omega
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 \\
  0 & L
\end{bmatrix}^{-1}
\begin{bmatrix}
  v_{fx} \\
  v_{fy}
\end{bmatrix} =
\begin{bmatrix}
  v_{fx} \\
  v_{fy}/L
\end{bmatrix}
\]

Bicycle Model
Example: Generalized Bicycle

• Models any vehicle whose wheels do not slip.

• Wheel position vectors:
  \[ \mathbf{\ell}_1^v = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^T \quad \mathbf{\ell}_2^v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}^T \]

• Skew matrix for wheel i:
  \[ \left[ \mathbf{\ell}_i^v \right]^\times = \begin{bmatrix} 0 & 0 & y_i \\ 0 & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \]

• For a single wheel (i):
  \[ v_i = \begin{bmatrix} (V_x - \omega y_i) \\ (V_y + \omega x_i) \end{bmatrix}^T \]
Example: Generalized Bicycle

- Models any vehicle whose wheels do not slip.

- Gather equations for both wheels:

\[
\begin{bmatrix}
    v_{1x} \\
    v_{1y} \\
    v_{2x} \\
    v_{2y}
\end{bmatrix} =
\begin{bmatrix}
    v_x - \omega y_1 \\
    v_y + \omega x_1 \\
    v_x - \omega y_2 \\
    v_y + \omega x_2
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & -y_1 \\
    0 & 1 & x_1 \\
    1 & 0 & -y_2 \\
    0 & 1 & x_2
\end{bmatrix}
\begin{bmatrix}
    v_x \\
    v_y \\
    \omega
\end{bmatrix}
\]

- Forward kinematics is again LPI:

\[
\dot{x}_y^w = [(H_c^v)^T (H_c^v)]^{-1} (H_c^v)^T v_c^w
\]
Example: 4 Wheel Steer (Inv)

- **Position vectors in body (vehicle) frame:**
  \[
  \mathbf{\ell}^v_{s1} = \begin{bmatrix} L \\ W \end{bmatrix}^T, \quad \mathbf{\ell}^v_{s2} = \begin{bmatrix} L \\ -W \end{bmatrix}^T, \quad \mathbf{\ell}^v_{s3} = \begin{bmatrix} -L \\ W \end{bmatrix}^T, \quad \mathbf{\ell}^v_{s4} = \begin{bmatrix} -L \\ -W \end{bmatrix}^T
  \]

- **Offset vectors in body frame:**
  \[
  \mathbf{\ell}^s_{c1} = d \begin{bmatrix} -s\gamma_1 \\ c\gamma_1 \end{bmatrix}^T, \quad \mathbf{\ell}^s_{c2} = d \begin{bmatrix} s\gamma_2 \\ -c\gamma_2 \end{bmatrix}^T, \quad \mathbf{\ell}^s_{c3} = d \begin{bmatrix} -s\gamma_3 \\ c\gamma_3 \end{bmatrix}^T, \quad \mathbf{\ell}^s_{c4} = d \begin{bmatrix} s\gamma_4 \\ -c\gamma_4 \end{bmatrix}^T
  \]

- **Equations for each wheel are:**
  \[
  \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\left( y_{si} + b_i \right) \\ 0 & 1 & \left( x_{si} + a_i \right) \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} -b_i \\ a_i \end{bmatrix} \gamma_i
  \]

NB: There are typos in the book for these eqns.

\[(4.62)\]
Example: 4 Wheel Steer (Inv)

- All together, these are of the form:

\[ \dot{\mathbf{y}}_c = H_c^w(\gamma) \dot{\mathbf{x}}_v + Q_c^w(\gamma) \dot{\gamma} \]

\[
\mathbf{H}_c^w(\gamma) = \begin{bmatrix}
1 & 0 & -(y_1 + b_1) \\
0 & 1 & x_1 + a_1 \\
1 & 0 & -(y_2 + b_2) \\
0 & 1 & x_2 + a_2 \\
1 & 0 & -(y_3 + b_3) \\
0 & 1 & x_3 + a_3 \\
1 & 0 & -(y_4 + b_4) \\
0 & 1 & x_4 + a_4 \\
\end{bmatrix}
\]

\[
Q_c^w(\gamma) = \begin{bmatrix}
-b_1 & 0 & 0 & 0 \\
0 & -b_2 & 0 & 0 \\
0 & a_2 & 0 & 0 \\
0 & 0 & -b_3 & 0 \\
0 & 0 & a_3 & 0 \\
0 & 0 & 0 & -b_4 \\
0 & 0 & 0 & a_4 \\
\end{bmatrix}
\]

- Forward kinematics is simple:

\[
\dot{\mathbf{x}}_v = [H_c^w(\gamma)^T H_c^w(\gamma)]^{-1} H_c^w(\gamma)^T [\mathbf{y}_c - Q_c^w(\gamma) \dot{\gamma}] 
\]

4 Wheel Steer
3D Case

- Works even if some vectors are out of the plane.

\[ \mathbf{v}_c^w = \mathbf{v}_v^w - \left[ \mathbf{r}_c^v \right] \times \mathbf{o}_v^w \]

Wheel Equation
Video
Outline

- 4.1 Moving Coordinate Systems
- 4.2 Kinematics of Wheeled Mobile Robots
  - 4.2.1 Aspects of Rigid Body Motion
  - 4.2.2 WMR Velocity Kinematics for Fixed Contact Point
  - 4.2.3 Common Steering Configurations
  - Summary
Summary

• The kinematic equations governing the motion of wheeled vehicles are those of planar rigid bodies.
  – It's all about the ICR.

• Rate kinematics for wheeled mobile robots are pretty straightforward
  – in the general case in 3D.

• The inverse problem is often overdetermined.
  – This is solved like any overdetermined system.